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Fractional Klein-Gordon equation with singular mass☆

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ABSTRACT

We consider a space-fractional wave equation with a singular mass term depending on the position and prove that it is very weak well-posed. The uniqueness is proved in some appropriate sense. Moreover, we prove the consistency of the very weak solution with classical solutions when they exist. In order to study the behaviour of the very weak solution near the singularities of the coefficient, some numerical experiments are conducted where the appearance of a wall effect for the singular masses of the strength of δ^2 is observed.

1. Introduction

In this work we investigate the Cauchy problem

$$\begin{cases} u_{tt}(t,x)+(-\Delta)^{\alpha}u(t,x)+m(x)u(t,x)=0, & (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where the spatially dependent coefficient m indicates the mass and the differential operator $(-\Delta)^{\alpha}$ stands for the fractional Laplacian. When $\alpha=1$ and the mass is constant, the equation in (1.1) reduces to the well known Klein-Gordon equation which plays a very important role in modelling many problems in classical and quantum mechanics, solitons and condensed matter physics.

Because of the non-local nature of the fractional derivatives, the fractional generalization of partial differential equations have been found to be very accurate to model real-world problems (see for instance [1–5]), while in the near past, they were thought to be the subject of the pure mathematics. Consequently, considerable attention has been given to the solution of fractional partial differential equations of physical interest. An important one of these equations is the fractional Klein-Gordon equation which has been generalized along two lines: one is to include a position dependent mass and the second is to use fractional derivatives instead of integer order derivatives. In the present paper, a space-fractional generalization of the Klein-Gordon equation with a position depen-

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dent mass is considered by replacing the classical Laplacian by the fractional one.

The study of analytical and numerical solutions of the space and/or time-fractional Klein-Gordon equation has been investigated considerably in the last years by many authors, we cite for instance [6–18] to mention only few of many recent publications. We also cite [19–22] where the authors consider the case when the mass term depends on the position and we refer to [23,24] where the fractional Laplacian is introduced. Our aim in this work is to contribute to the study of the well-posedness of the Cauchy problem (1.1) where we allow the spatially dependent coefficient m to be singular. This needs to give a meaningful notion of solution, since in the case of distributional data, the multiplication in the equation in (1.1) doesn't make sense (see [25]) and thus the problem can not be formulated. In this context, the authors in [26] introduced the concept of very weak solutions to the study of second order hyperbolic equations with irregular coefficients and was later applied in [27–29] for different physical models. We want here to apply it for our considered model in order to show its wide applicability. We note here that our intention to consider irregular coefficients is physically motivated by the fact that in the microscopic scale, e.g. in the theory of fluids, the mass behaves like distributions [30].

As mentioned above we want to study the well-posedness in the very weak sense of the Cauchy problem (1.1). The uniqueness is proved in an appropriate sense. Moreover, we prove the consistency of the very weak solution with the classical ones when they exist. The mass coefficient is spatially dependent, it is our task to do numerical experiments in order to study the behaviour of the very weak solution near the singularities of the coefficient.

The leading objective of the present investigation is to contribute to the study of the well-posedness of the the space-fractional Klein-Gordon equation with a singular mass term using the recently introduced concept of very weak solutions. The advantage in using this concept is that we are allowed to consider coefficients with strong singularities, e.g. the Dirac delta function and its powers. Another aim is to show that the concept is easy to use in applications.

2. Main results

For $\alpha > 0$ and $d \in \mathbb{N}$, we investigate the Cauchy problem for the fractional Klein-Gordon equation

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\alpha}u(t,x) + m(x)u(t,x) = 0, \ \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), \ \ x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

Here, the function m is supposed to be non-negative and singular. In the regular situation, i.e. in the case when the coefficient m is a regular function we have the following lemma.

To start with, let us define some notions and notations that we use throughout this paper. Firstly, the notation $f \lesssim g$ means that there exists a positive constant C such that $f \leq Cg$. Secondly, the fractional Sobolev space $H^{\alpha}(\mathbb{R}^d)$ is defined as follows : $H^{\alpha}(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : \|u\|_{H^{\alpha}} < +\infty\right\}$, where $\|u\|_{H^{\alpha}} = \|u\|_{L^2} + \|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}$ and by $(-\Delta)^{\alpha}$ we mean the standard definition of the fractional Laplacian in terms of the Fourier transform, that is, $(-\Delta)^{\alpha}u = \mathcal{F}^{-1}(|\xi|^{2\alpha}(\mathcal{F}u))$ for all $\xi \in \mathbb{R}^d$. We refer to [31] and [32] for more details and alternative definitions of the fractional Laplacian and the fractional Sobolev spaces.

We will also use the following notation:

$$\|u(t,\cdot)\| := \|u(t,\cdot)\|_{H^{\alpha}} + \|\partial_t u(t,\cdot)\|_{L^2}.$$

Lemma 1. *Let $m \in L^{\infty}(\mathbb{R}^d)$ and $m \geq 0$. Suppose that $u_0 \in H^{\alpha}(\mathbb{R}^d)$ and $u_1 \in L^2(\mathbb{R}^d)$. Then, there is a unique solution $u \in C([0,T]; H^{\alpha}(\mathbb{R}^d)) \cap C^1([0,T]; L^2(\mathbb{R}^d))$ to (2.1), and it satisfies the estimate*

$$\|u(t,\cdot)\|^2 \lesssim (1 + \|m\|_{L^{\infty}})\left[\|u_1\|_{L^2}^2 + \|u_0\|_{H^{\alpha}}^2\right]. \tag{2.2}$$

Proof of Lemma 1.. Multiplying the Eq. (2.1) on both sides by u_t and integrating, we get

$$\begin{aligned} &Re\langle\partial_t^2 u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} + Re\langle(-\Delta)^{\alpha}u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} \\ &+Re\langle m(\cdot)u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = 0, \end{aligned} \tag{2.3}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ is the inner product of $L^2(\mathbb{R}^d)$.

Easy calculations show that

$$Re\langle\partial_t^2 u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\langle\partial_t u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2},$$

$$Re\langle(-\Delta)^{\alpha}u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\langle(-\Delta)^{\frac{\alpha}{2}}u(t,\cdot), (-\Delta)^{\frac{\alpha}{2}}u(t,\cdot)\rangle_{L^2},$$

and

$$Re\langle m(\cdot)u(t,\cdot), \partial_t u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t\langle m^{\frac{1}{2}}(\cdot)u(t,\cdot), m^{\frac{1}{2}}(\cdot)u(t,\cdot)\rangle_{L^2}.$$

Let us denote by

$$E(t) := \|\partial_t u(t,\cdot)\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}}u(t,\cdot)\|_{L^2}^2 + \|m^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2,$$

the energy functional of the system (2.1). From (2.3) it follows that $\partial_t E(t) = 0$, and thus $E(t) = E(0)$. By taking in consideration that $\|m^{\frac{1}{2}}u_0\|_{L^2}^2$ can be estimated by $\|m^{\frac{1}{2}}u_0\|_{L^2}^2 \leq \|m\|_{L^{\infty}}\|u_0\|_{L^2}^2$, it follows that

$$\|\partial_t u(t,\cdot)\|_{L^2}^2 \lesssim \left(\|u_1\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}}u_0\|_{L^2}^2 + \|m\|_{L^{\infty}}\|u_0\|_{L^2}^2\right), \tag{2.4}$$

$$\|(-\Delta)^{\frac{\alpha}{2}}u(t,\cdot)\|_{L^2}^2 \lesssim \left(\|u_1\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}}u_0\|_{L^2}^2 + \|m\|_{L^{\infty}}\|u_0\|_{L^2}^2\right), \tag{2.5}$$

and

$$\|m^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \lesssim \left(\|u_1\|_{L^2}^2 + \|(-\Delta)^{\frac{\alpha}{2}}u_0\|_{L^2}^2 + \|m\|_{L^{\infty}}\|u_0\|_{L^2}^2\right). \tag{2.6}$$

Hence, the desired estimates for $\partial_t u(t,\cdot)$ and $(-\Delta)^{\frac{\alpha}{2}}u(t,\cdot)$ are proved. Let us now estimate u. Applying the Fourier transform to (2.1), the problem can be rewritten as a second order ordinary differential equation

$$\hat{u}_{tt}(t,\xi) + |\xi|^{2\alpha}\hat{u}(t,\xi) = \hat{f}(t,\xi), \tag{2.7}$$

with the initial conditions $\hat{u}(0,\xi) = \hat{u}_0(\xi)$ and $\hat{u}_t(0,\xi) = \hat{u}_1(\xi)$. Here $\hat{f}$, $\hat{u}$, denote the Fourier transform of f and u in the spacial variable and $f(t,x) := -m(x)u(t,x)$. We note that in (2.7), we see $\hat{f}$ as a source term.

By solving first the homogeneous equation and by application of Duhamel's principle (see, e.g. [33]), we get the following representation of the solution

$$\begin{aligned} \hat{u}(t,\xi) &= \cos(t|\xi|^{\alpha})\hat{u}_0(\xi) + \frac{\sin(t|\xi|^{\alpha})}{|\xi|^{\alpha}}\hat{u}_1(\xi) \\ &+ \int_0^t \frac{\sin((t-s)|\xi|^{\alpha})}{|\xi|^{\alpha}}\hat{f}(s,\xi)ds. \end{aligned} \tag{2.8}$$

Taking the L^2 norm in (2.8) and using the following estimates: 1) $|\cos(t|\xi|^{\alpha})| \leq 1$, for $t \in [0,T]$ and $\xi \in \mathbb{R}^d$, 2) $|\sin(t|\xi|^{\alpha})| \leq 1$, for large frequencies and $t \in [0,T]$ and, 3) $|\sin(t|\xi|^{\alpha})| \leq t|\xi|^{\alpha} \leq T|\xi|^{\alpha}$, for small frequencies and $t \in [0,T]$, we get that

$$\|\hat{u}(t,\cdot)\|_{L^2}^2 \lesssim \|\hat{u}_0\|_{L^2}^2 + \|\hat{u}_1\|_{L^2}^2 + \int_0^t \|\hat{f}(s,\cdot)\|_{L^2}^2 ds.$$

By Parseval-Plancherel formula we arrive at

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \int_0^T \|m(\cdot)u(s,\cdot)\|_{L^2}^2 ds.$$

Using the estimate (2.6) and taking in consideration that the last term in the above estimate can be estimated by $\|m(\cdot)u(t,\cdot)\|_{L^2} \le \|m\|_{L^\infty}^{\frac{1}{2}} \|m^{\frac{1}{2}} u(t,\cdot)\|_{L^2}$, we get

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim (1+\|m\|_{L^\infty})\left[\|u_0\|_{H^\alpha}^2 + \|u_1\|_{L^2}^2\right]. \tag{2.9}$$

The estimate (2.2) follows by summing the estimates (2.4), (2.5) and (2.9), ending the proof. □

2.1. Very weak solutions: existence

Here, we consider an irregular case when the mass term m of the Eq. (2.1) has strong singularities, namely, δ-function or "δ^2-function" like behaviours. In what follows, we will understand a multiplication of distributions in the sense of the Colombeau algebra [34].

Now we introduce a notion of the very weak solution to the Cauchy problem (2.1) and prove the existence result. We start by regularising the coefficient m using a suitable mollifier ψ generating families of smooth functions $(m_\varepsilon)_\varepsilon$, namely, $m_\varepsilon(x) = m * \psi_\varepsilon(x)$, where $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$ and $\varepsilon \in (0,1]$. The function ψ is a Friedrichs-mollifier, i.e. $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \ge 0$ and $\int \psi = 1$.

Assumption 1. We make the following assumption on the regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m: there exist $N_0 \in \mathbb{N}_0$ and $C > 0$ such that

$$\|m_\varepsilon\|_{L^\infty} \le C\varepsilon^{-N_0}, \tag{2.10}$$

for all $\varepsilon \in (0,1]$.

We note that by the structure theorems of distributions, such assumption is natural and is satisfied, e.g, for $m \in \mathcal{D}'$. Let us give some examples.

Example 1. Let $m(x) = \delta_0(x)$. Then, we have $m_\varepsilon(x) = m * \psi_\varepsilon(x) = \varepsilon^{-d}\psi(\varepsilon^{-1}x) \le C\varepsilon^{-d}$. Moreover, for $m(x) = \delta_0^2(x)$, one can define $m_\varepsilon(x) = \varepsilon^{-2d}\psi^2(\varepsilon^{-1}x) \le C\varepsilon^{-2d}$.

Definition 1 (Moderateness)**.**

(i) A net of functions $(g_\varepsilon)_\varepsilon$ is said to be L^∞-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|g_\varepsilon\|_{L^\infty} \le c\varepsilon^{-N}.$$

(ii) A net of functions $(u_\varepsilon)_\varepsilon$ from $C([0,T];H^\alpha) \cap C^1([0,T];L^2)$ is said to be C^1-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\| \le c\varepsilon^{-N}.$$

Remark 1. By the Assumption (2.10), m_ε is L^∞-moderate in the sense of the last definition.

Definition 2 (Very Weak Solution)**.** Let $(u_0,u_1) \in H^\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then the net $(u_\varepsilon)_\varepsilon \in C([0,T];H^\alpha(\mathbb{R}^d)) \cap C^1([0,T];L^2(\mathbb{R}^d))$ is a very weak solution to the Cauchy problem (2.1) if there exists an L^∞-moderate regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m such that $(u_\varepsilon)_\varepsilon$ solves the regularized problem

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + (-\Delta)^\alpha u_\varepsilon(t,x) + m_\varepsilon(x)u_\varepsilon(t,x) = 0, & (t,x) \in (0,T]\times\mathbb{R}^d, \\ u_\varepsilon(0,x) = u_0(x),\ \partial_t u_\varepsilon(0,x) = u_1(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.11}$$

for all $\varepsilon \in (0,1]$, and is C^1-moderate.

Theorem 2. *Assume that the regularisation $(m_\varepsilon)_\varepsilon$ of the coefficient m satisfies the moderateness condition (2.10). Then the Cauchy problem (2.1) has a very weak solution.*

Proof of Theorem 2.. Since u_0 and u_1 are smooth enough, using the moderateness Assumption (2.10) and the energy estimate (2.2), we arrive at

$$\|u_\varepsilon\| \le C\varepsilon^{-N_0/2},$$

where N_0 is from (2.10), which means that $(u_\varepsilon)_\varepsilon$ is C^1-moderate. □

2.2. Uniqueness

The uniqueness of the very weak solution is proved in the sense of the following definition.

Definition 3. We say that the Cauchy problem (2.1) has a unique very weak solution, if for all families of regularisations $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$, of the coefficient m, satisfying $\|m_\varepsilon - \tilde{m}_\varepsilon\|_{L^\infty} \le C_k\varepsilon^k$ for all $k > 0$, it follows that

$$\|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \le C_N\varepsilon^N$$

for all $N > 0$, for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions corresponding to $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$, respectively.

Theorem 3. *Let $T > 0$. Assume that $m \ge 0$ in the sense that its regularisations as functions are non-negative. Suppose that $(u_0,u_1) \in H^\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then, the very weak solution to the Cauchy problem (2.1) is unique.*

Proof of Theorem 3.. Let $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ be very weak solutions to the Cauchy problem (2.1) corresponding to the coefficients $(m_\varepsilon)_\varepsilon$ and $(\tilde{m}_\varepsilon)_\varepsilon$ and assume that $\|m_\varepsilon - \tilde{m}_\varepsilon\|_{L^\infty} \le C_k\varepsilon^k$ for all $k > 0$. Let us denote by $U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x)$, then, U satisfies the equation

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x) + (-\Delta)^\alpha U_\varepsilon(t,x) + m_\varepsilon(x)U_\varepsilon(t,x) = f_\varepsilon(t,x), \\ U(0,x) = 0, \quad \partial_t U_\varepsilon(0,x) = 0, \end{cases} \tag{2.12}$$

with $f_\varepsilon(t,x) = (\tilde{m}_\varepsilon(x) - m_\varepsilon(x))\tilde{u}_\varepsilon(t,x)$. Using Duhamel's principle, U_ε is given by $U_\varepsilon(x,t) = \int_0^t V_\varepsilon(x,t-s;s)ds$, where $V_\varepsilon(x,t;s)$ solves the problem

$$\begin{cases} \partial_t^2 V_\varepsilon(x,t;s) + (-\Delta)^\alpha V_\varepsilon(x,t;s) + m_\varepsilon(x)V_\varepsilon(x,t;s) = 0, \\ V_\varepsilon(x,0;s) = 0,\ \ \partial_t V_\varepsilon(x,0;s) = f_\varepsilon(s,x). \end{cases}$$

Taking U_ε in L^2-norm and using (2.2) to get estimate for V_ε, we arrive at

$$\begin{aligned} \|U_\varepsilon(\cdot,t)\|_{L^2} &\le C(1+\|m_\varepsilon\|_{L^\infty})^{\frac{1}{2}} \int_0^T \|f_\varepsilon(s,\cdot)\|_{L^2}ds \\ &\le C(1+\|m_\varepsilon\|_{L^\infty})^{\frac{1}{2}} \|\tilde{m}_\varepsilon - m_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2}ds. \end{aligned}$$

We have that $\|m_\varepsilon - \tilde{m}_\varepsilon\|_{L^\infty} \le C_k\varepsilon^k$ for all $k > 0$, the net $(m_\varepsilon)_\varepsilon$ is moderate by assumption and $(\tilde{u}_\varepsilon)_\varepsilon$ is moderate as a very weak solution to the Cauchy problem (2.1). Then, for all $N > 0$, we obtain

$$\|U_\varepsilon(\cdot,t)\|_{L^2} = \|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^N.$$

Thus, the very weak solution is unique. □

2.3. Consistency

We want to prove that in the case when a classical solution exists for the Cauchy problem (2.1) as in Lemma 1, the very weak solution recaptures the classical one.

Theorem 4. *Let $(u_0,u_1) \in H^\alpha(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Assume that $m \in L^\infty(\mathbb{R}^d)$ is non-negative and, let us consider the Cauchy problem*

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\alpha u(t,x) + m(x)u(t,x) = 0, & (t,x) \in (0,T]\times\mathbb{R}^d, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.13}$$

Let $(u_\varepsilon)_\varepsilon$ *be a very weak solution of (2.13). Then for any regularising family* $m_\varepsilon = m * \psi_\varepsilon$, *for any* $\psi \in C_0^\infty$, $\psi \geq 0$, $\int \psi = 1$, *the net* $(u_\varepsilon)_\varepsilon$ *converges to the classical solution of the Cauchy problem (2.13) in* L^2 *as* $\varepsilon \to 0$.

Proof of Theorem 4.. The classical solution satisfies

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\alpha u(t,x) + m(x)u(t,x) = 0, & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x),\, u_t(0,x) = u_1(x), & x \in \mathbb{R}^d. \end{cases}$$

For the very weak solution, there is a representation $(u_\varepsilon)_\varepsilon$ such that

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + (-\Delta)^\alpha u_\varepsilon(t,x) + m_\varepsilon(x) u_\varepsilon(t,x) = 0, & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_0(x),\, \partial_t u_\varepsilon(0,x) = u_1(x), & x \in \mathbb{R}^d. \end{cases}$$

Taking the difference of the above equations, we get

$$\begin{cases} \partial_t^2 (u-u_\varepsilon)(t,x) + (-\Delta)^\alpha (u-u_\varepsilon)(t,x) + m_\varepsilon(x)(u-u_\varepsilon)(t,x) = \eta_\varepsilon(t,x), \\ (u-u_\varepsilon)(0,x) = 0, \quad \partial_t (u-u_\varepsilon)(0,x) = 0, \quad x \in \mathbb{R}^d, \end{cases}$$

where $\eta_\varepsilon(t,x) = (m(x) - m_\varepsilon(x))u(t,x)$. Let us denote by $W_\varepsilon(t,x) := (u-u_\varepsilon)(t,x)$. Once again, using Duhamel's principle, W_ε is given by $W_\varepsilon(x,t) = \int_0^t V_\varepsilon(x, t-s; s)ds$, where $V_\varepsilon(x,t;s)$ solves the problem

$$\begin{cases} \partial_t^2 V_\varepsilon(x,t;s) + (-\Delta)^\alpha V_\varepsilon(x,t;s) + m_\varepsilon(x) V_\varepsilon(x,t;s) = 0, \\ V_\varepsilon(x,0;s) = 0, \; \partial_t V_\varepsilon(x,0;s) = \eta_\varepsilon(s,x). \end{cases}$$

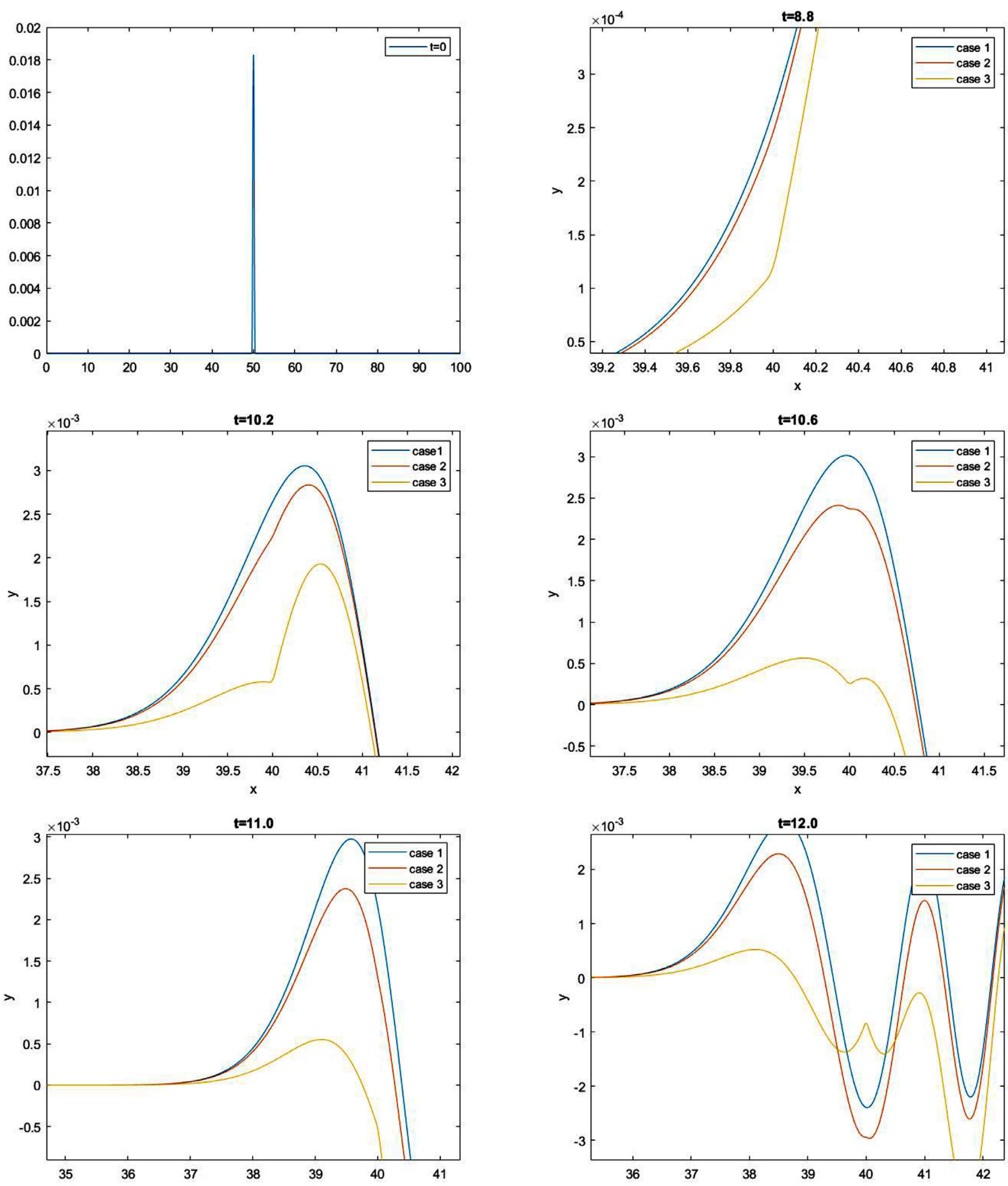

Fig. 1. In these plots, we analyse behaviours of the solutions of the Eq. (3.1) in the cases of different mass terms. In the upper-left plot, the graphic of the initial function u_0 is given. In the further plots, we compare the replacement function u at $t = 8.8, 10.2, 10.6, 11.0, 12.0$ for $\varepsilon = 0.05$ in the three cases of the mass term, which are described below.

We have that $\|m - m_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0$. Taking the L^2-norm for W_ε and using the energy estimate (2.2), we get

$$\|W_\varepsilon(\cdot,t)\|_{L^2} \le \int_0^T \|V_\varepsilon(\cdot,t-s;s)\|_{L^2} ds \\ \le C(1+\|m_\varepsilon\|_{L^\infty})^{1/2}\|m-m_\varepsilon\|_{L^\infty}^{1/2} \int_0^T \|u(s,\cdot)\|_{L^2} ds.$$

Since $\|m_\varepsilon\|_{L^\infty} \le C$ it follows that $(u_\varepsilon)_\varepsilon$ converges to u in L^2 as $\varepsilon \to 0$. $\square$

3. Numerical experiments

In this Section, we do some numerical experiments. We note that in the case when the mass m depends only on the parameter t, the simulations were done in [35]. Let us analyse our problem by regularising a distributional mass term $m(x)$ by a parameter ε. We define $m_\varepsilon(x) := (m * \varphi_\varepsilon)(x)$, as the convolution with the mollifier $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$, where $\varphi(x) = \begin{cases} c\exp\left(\frac{1}{x^2-1}\right), & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$ with $c \simeq 2.2523$ to have $\int\limits_{-\infty}^{\infty} \varphi(x)dx = 1$. Then, instead of (2.1) we consider the regularised problem

$$\partial_t^2 u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) + m_\varepsilon(x)u_\varepsilon(t,x) = 0, \ (t,x) \in (0,T] \times \mathbb{R}, \tag{3.1}$$

with the initial data $u_\varepsilon(0,x) = u_0(x)$ and $\partial_t u_\varepsilon(0,x) = u_1(x)$, for all $x \in \mathbb{R}$. Here, we put

$$u_0(x) = \begin{cases} \exp\left(\frac{1}{(x-50)^2-0.25}\right), & |x-50| < 0.5, \\ 0, & |x-50| \ge 0.5, \end{cases}$$

and $u_1(x) \equiv 0$. Note that $\operatorname{supp} u_0 \subset [49.5, 50.5]$.

For m we consider the following cases, with δ denoting the standard Dirac's delta-distribution:

Case 1: $m(x) = 0$ with $m_\varepsilon(x) = 0$;
Case 2: $m(x) = \delta(x-40)$ with $m_\varepsilon(x) = \varphi_\varepsilon(x-40)$;
Case 3: $m(x) = \delta(x-40) \times \delta(x-40)$. Here, we understand $m_\varepsilon(x)$ as $m_\varepsilon(x) = (\varphi_\varepsilon(x-40))^2$.

In Fig. 1, we analyse behaviours of the solutions to the Eq. (3.1) with the initial function u_0 (given in the upper-left plot) in the cases of different mass terms. The further plots of Fig. 1 are comparing the replacement function u at $t = 8.8, 10.2, 10.6, 11.0, 12.0$ for $\varepsilon = 0.05$ in the following three cases: Case 1 is corresponding to the mass term m is equal to zero; Case 2 is corresponding to the case when the mass term m is like a δ-function; Case 3 is corresponding to the mass term m is like a square of the δ-function.

By analysing Fig. 1, we see that a delta-function mass term affects less on the behaviour of the solution of (3.1) compared to the square delta-function like mass term by reflecting some waves in the opposite direction. In the upper-right plot and in the lower plots of Fig. 1, we observe that the replacement function u is almost fully reflected in the square delta-function like mass term case. At $t = 8.8$ we see that the yellow coloured wave is starting to settle and, from $t = 10.2$ is moving in opposite direction. We call the last phenomena, a "wall effect".

All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take $\Delta t = 0.2$, $\Delta x = 0.01$.

4. Conclusion

The analysis conducted in this article shows that numerical methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The concept of very weak solutions eliminates this difficulty in the case of the terms with multiplication of distributions. In contrast with the framework of the Colombeau algebras (see [34]) where the consistency with classical solutions maybe lost, the concept of very weak solutions which depends heavily on the equation under consideration is consistent with classical theory. In particular, in the Klein-Gordon equation case, we see that a delta-function mass term affects less on the behaviour of the waves compared to the square of the delta-function case, the latter causing a so-called "wall effect".

Numerical experiments have shown that the concept of very weak solutions is very suitable for numerical modelling. In addition, using the theory of very weak solutions, we can talk about the uniqueness of numerical solutions of differential equations with strongly singular coefficients in an appropriate sense.

Essentially, the present work can be considered as a generalization of the study of the Klein-Gordon equation by introducing the fractional Laplacian instead of the classical one and by considering a spatially dependent mass. Moreover, we are treating the case of singular masses which has been less investigated in the literature.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Arshyn Altybay: Visualization, Methodology. **Michael Ruzhansky:** Supervision, Investigation. **Mohammed Elamine Sebih:** Investigation, Writing - review & editing. **Niyaz Tokmagambetov:** Investigation, Writing - review & editing.

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